CHAPTER VI

CSHETRA-VYAVAHARA[1]
(Determination of plane figure[2])

RECTANGULAR TRIANGLE
(Right-angled Triangle)

133) Naming two legs of a rectangular triangle

A bhuja (side) is assumed. The other side, in the rival direction, is called the coti (upright), whether in a triangle or tetragon, by persons conversant with the subject.

[1] Cshetra - plane surface, bounded by a figure; as triangle, &c.
Vyavahara - ascertainment of its dimensions, as diagonal, perpendicular, area, &c. - GANESA

[2] Plane figure is four-fold: triangle, quadrangle, circle and bow.

Triangle is a figure containing'*tri*' '*asra*' (or *cona*), (angles); - *tryasra* or *tricona* and containing as many '*bhujas*' (arms) - *tribhuja*]

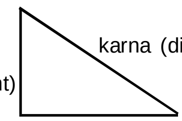
Quadrangle - *chaturasra, chaturcona, chaturbhuja.* These are two fold.
1) *SAMA-CARNA* (equal diagonals):
suare, trapezium, oblique parallelogram and rectangle

2) *VISHAMA-CARNA* (unequal diagonals):
(i) rhombus (ii) 3 equal sides (iii) 2 pairs of equal sides (iv) 2 sides equal (v) 4 unequal sides (vi) equal perpendiculars.

Circle, Bow need no definition - GANESA

<u>AUTHOR'S NOTE</u>:

Vedic *rituals* needed *exact* square and circular pits; more than one pit (simultaneously) with specific relative positions, dimensions etc. for the '*sacrificial fires*'. So the necessary skills (regarding squares, triangles, circles etc.) existed, in the ***vedic period*** itself.



Obviously the Rishis knew the facts connected with these, they need not had to prove it to anyone else. Labourers were given the necessary techniques; not the theory. In those days any 'knowledge' was preserved within families down the generations. It was a 'life-long apprenticeship'. The division of labour was the 'perfect' system. When economic disparities crept it, obviously, the system lost its value and itself became a problem. Recording methods, procedures, etc. came much later.

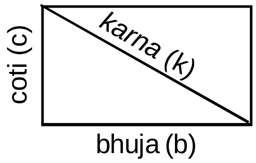
In *Sluvasutras* (800 B.C.), we find many different geometrical facts and procedures. The equipment they had were straightened bamboo's and 'treated' rope. They have used, among other things, the fact that 3, 4, 5; 5, 12, 13; ... (which, we now call Pythagorean triplets) made a right-corners (angles).

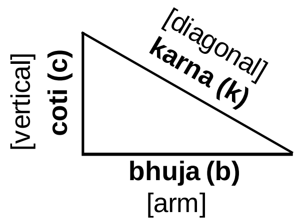
Coming to this right-angled triangle, almost every society had explored it along with circle, square, diagonal, ... Babyonians (3000 B.C.) had accurately recorded the diagonal of a 1 x 1 square, correct to 4 decimal places.

Bhascara gives two easy short-cut methods for performing the calculations to find the third side when we know two sides of the rectangle triangle. But he also gives many more rules to find the unknown sides under very many different situations. And all these are done mentally. Today, any of us would need paper and pen and 'time' to work these out.

Bhascara is a great poet too. So, his problems are simply beautiful and natural. I am sure every one will enjoy the next few pages even more than the others.

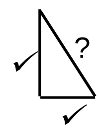
134) Relationships between legs & karna[1] ...



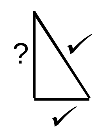


1) *Bhuja, Coti given*:
The square-root of the sum of the squares of those legs is the karna.[2]

$$\text{karna} = \sqrt{\text{bhuja}^2 + \text{coti}^2}$$

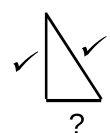


2) *Karna, Bhuja given*:
The square-root, extracted from the difference of the squares of the karna and side, is the coti (upright):



$$\text{coti} = \sqrt{\text{karna}^2 - \text{bhuja}^2}$$

3) *Karna, Coti given*:
and that, extracted from the difference of the squares of the karna and upright, is the bhuja (side).[3]



$$\text{bhuja} = \sqrt{\text{karna}^2 - \text{coti}^2}$$

[1] *Karna* - A thread or oblique line from the two extremities of the *legs*, joining them; also termed *sruti*, *sravana*, or by any other words importing ear.

In the case of a triangle, it is the diagonal of the parallelogram, whereof the triangle is the half: and is the hypotenuse of a right-angled triangle.

[2] These results were used in India since the time of *Sulvasutrakaras (*B.C. 3000 - 800); whereas Pythgoras result was published, apparently, in 560 B.C.

[3] The proof is given both algebraically and geometrically by GANESA.
BHASCARA has himself given a demonstration of the rule in his *algebraical* work, *Vija Ganita* §146.

135) Two Identities ... - Sum of Squares of 2 Quntities ...

Sanskrit ‘ya’ is the initial syllable for ‘yavat’ and sanskrit ‘ru’ is the initial syllable for ‘rupa’. These are like the ‘x’, ‘y’ we use in algebra today!

As I am having difficulty with Sanskrit scripts, I shall use ‘*y*’ and ‘*r*’ for now.

1) *Twice the product of two quantities, added to the square of their difference will be the sum of their squares.*

$$y^2 + r^2 = (y - r)^2 + 2\,yr$$

Examples:

1) 13, 5 are two quantities.

$13^2 + 5^2 = (diff)^2 + 2$ prod, $(13-5)^2 + 2(13x5)$, $8^2 + 2(65)$, **194**

2) 12, 17 are two quantities.

$12^2 + 7^2 = 5^2 + 2(84)$, **193**

3) 21, 9 are two quantities.

$21^2 + 9^2 = 12^2 + 2(189)$, **522**

In today’s terminology: $a^2 + b^2 = (a - b)^2 + 2ab$

Note: You may appreciate the method, not with these small numbers but with large numbers (specially in astronomical calculations)

135) ... Two Identities - Difference of Squares of 2 Quntities

2) *The product of their sum and difference will be the difference of their squares: as must be every where understood by the intelligent calculator.*

$$y^2 - r^2 = (y + r)(y - r)$$

Example:

1) 15, 12 are two quantities.

$15^2 - 12^2 = (sum)(diff) = 27 \times 3 = 81$

2) 22, 18 are two quantities.

$22^2 - 18^2 = 40 \times 4$, **160**

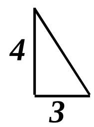
3) 75, 25 are two quantities.

$75^2 - 25^2 = 100 \times 50$, **5000**

In today's terminology: $a^2 - b^2 = (a - b)(a + b)$

136) Example ...

Where the coti is four and the bhuja three, what is the karna? ...

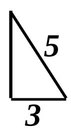


Statement: bhuja 3; coti 4.

• Sum of squares 25 (or)
• Double prod of sides, 24; Sq of diff 1; added 25.

Karna, sq rt of this 25, 5

136)   ... Tell me also the coti from the karna and bhuja; ...

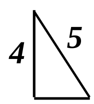


Statement: bhuja 3; karna 5.

• Diff of squares 16 (or)
• Product of sum & diff, 8 x 2, 16

Coti, sq rt of this 16, 4

136)   ... and the bhuja from the coti and karna.



Statement: coti 4; carna 5.

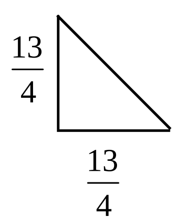
• Diff of squares 9 (or)
• Product of sum & diff, 9 x 1, 9

Bhuja, sq rt of this 9, 3

137) Example

Where the bhuja measures three and a quarter; and the coti as much; tell me, quickly, mathematician, what is the length of the karna?

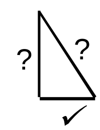
Statement:



Twice their product + sq of diff, $\frac{169}{8}$; Carna, sq rt of $\frac{169}{8}$.

<u>139) *Bhuja given*</u>

A bhuja is put. From that multiplied by twice some assumed number, and divided by one less than the square of the assumed number, a coti is obtained. This, being set apart, is multiplied by the arbitrary number, and the bhuja as put is subtracted; the remainder will be the karna. (Results are rational quantities)



<u>GIVEN</u>: *bhuja* (*b*), Assume a number (*n*), Coti ? Karna ?

COTI:
- Multiply bhuja by twice assumed number
- Divide this by '1 less than sq of assumed number'

KARNA:
- Multiply coti by assumed number
- Subtract bhuja

<u>141) Example</u> Bhuja is 12. Give many cotis and karnas.

(1) Assume 2:

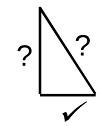
COTI:	[bhuja x 2(assm. no.)] / [assm. no.^{2} – 1]	48/3	16
KARNA:	[coti x assm. no.] – *bhuja:*	32 – 12	20

Coti, 16; Karna, 20

(2)	Assume 3:	Coti:	72/8 =	9
		Karna:	27 – 12 =	15
(3)	Assume 5:	Coti:	120/24 =	5
		Karna:	25 – 12 =	13
(4)	Assume 4:	Coti:	96/15 =	6.4
		Karna:	25.6 – 12 =	13.6

140) *Bhuja given* (alternative method)

Or a bhuja is put. Its square, divided by an arbitrary number, is set down in two places: and the arbitrary number being added and subtracted, and the sum and difference halved, the results are the karna and coti. (Results are rational quantities)



Or, in like manner, the bhuja and karna may be deduced from the coti.

<u>GIVEN</u>: *bhuja* (*b*), Assume a number (*n*), Coti ? Karna ?

COTI:
- Divide sq of bhuja by assumed number
- Subtract assumed number and halve the result

KARNA:
- In the 2nd step above, add instead of 'subtract'.

<u>141) Example</u> Bhuja is 12. Give many cotis and karnas.

(1) Assume 2:

COTI:	[bhuja^2 / assm. no.] – assm. no.; halve it	[72 – 2]/2	35
KARNA:	'Add' instead of 'subtract'	[72 + 2]/2	37

Coti, 16; Karna, 20

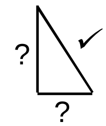
(2)	Assume 3:	Coti:	48 – 3	22.5
		Karna:		25.5

[Note that, in this method, karna is coti + assm. no.!]

(3)	Assume 4:	Coti:	36 – 4	16
		Karna:		20

142) *Karna given*

Twice the karna taken into an arbitrary number, being divided by the square of the arbitrary number added to one, the quotient is the coti. This taken apart is to be multiplied by the number put: the difference between the product and the karna is the bhuja. (Rational numbers)



GIVEN: *karna*; Choose an arbitrary number, Coti ?, Bhuja ?

COTI:
- Multiply 'twice karna' by assumed number
- Divide by 'sq. of assm. no.' increased by 1

BHUJA:
- Multiply coti by assm. no.; Diff with Karna

143) Example

Karna being measured by eighty-five, say promptly, learned man, what cotis and bhujas will be rational?

Statement: karna 85, coti ?, bhuja ?

(1) Assume 2:

COTI:	[twice karna x assm. no.] / [assm. no.^{2} + 1]	340 / 5	68
BHUJA:	Diff. between [coti x assm. no.] and karna	136 – 85	51

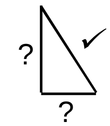
Coti, 68; Bhuja, 51

(2) Assume 4:

Coti:	680 / 17	40
Karna:	160 – 85	75

144) ***Karna given*** (alternative method)

Or else karna is doubled and divided by the square of an assumed number added to one. Karna, less that quotient, is the coti. The same quotient, multiplied by the assumed number, is the bhuja.



GIVEN: *karna*; Choose an arbitrary number, Coti ?, Bhuja ?

COTI:
- Divide 'twice karna' by 1 more than sq. of assumed number
- Subtract from karna

BHUJA:
- Instead of 2nd step above, multiply by assm. no.

143) Example

Karna being measured by eighty-five, say promptly, learned man, what cotis and bhujas will be rational?

Statement: karna 85, coti ?, bhuja ?

(1) Assume 2:

COTI:	[2 x karna] / [assm. no.^{2} + 1]; Subt. from karna	85 – 34	41
BHUJA:	Do not subtr. but multiply be assm. no.	34 x 2	68

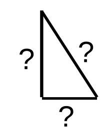
Coti, 51; Bhuja, 68

(2) Assume 4:

Coti:	85 – 10	75
Karna:	10 x 4	40

145) *NONE* given

Let twice the product of two assumed numbers be the coti; and the difference of their squares, the bhuja: the sum of their squares will be the carna, and a rational number.



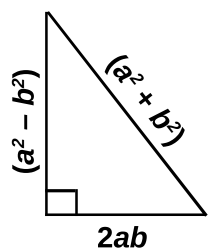
GIVEN: Assume two numbers (a, b)

Coti? *TWICE* their *product*

Bhuja? *DIFFERENCE* of their squares

Karna? *SUM* of their squares

$2ab$
$a^2 - b^2$
$a^2 + b^2$



assumed nos a b	BHUJA twice their prod. $2ab$	COTI diff of sqs. $(a^2 - b^2)$	CARNA sum of sqs $(a^2 + b^2)$
1, 2	4	3	5
2, 3	12	5	13
2, 4	16	12	20
3, 7	42	40	58
11, 35	770	1104	1346

Note: You may like to see author's ***Quick Maths*** or ***Magic Maths*** for a very detailed and comprehensive work on generating distinct sets of sides of rectangular triangles (triplets) easily in one step.

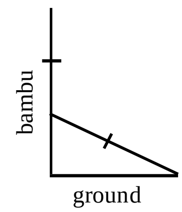
147) Bhuja & Sum (of other arms) given

The square of the ground intercepted between the root[1] and tip, is divided by the [length of the] bambu; and the quotient severally added to, and subtracte from, the bambu: the moieties [of the sum and difference] will be the two portions of it representing karna and coti.

[1] (A tall vertical bambu breaks at some point and the tip touches the ground)

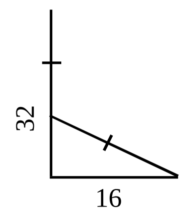
GIVEN: bhuja (ground), sum of coti and karna (bambu)

Divide ground^2 by bambu;
- Halve sum with *bambu* (Karna)
- Halve diff with *bambu* (Coti)



148) Example

If a bambu measuring 32 cubits and standing upon level ground, be broken in one place, by the force of the wind, and the tip of it meet the ground at 16 cubits: say, mathematician, at how many cubits from the root is it broken?

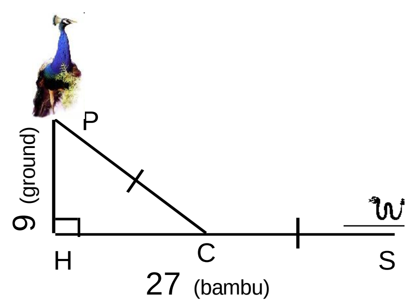


Given: bambu 32, ground 16, coti ?

Divide sq of ground by bambu, 8; Half (32 – 8), 12;
Bambu broke at **12** cubits from the root.

150) Example

A snake's hole is at the foot of a pillar, 9 cubits tall and a peacock is perched on its summit. Seeing a snake, at the distance of thrice the pillar, gliding towards his hole, he pounces obliquely upon him. Say quickly at how many cubits from the snake's hole do they meet, both proceeding an equal distance?



Divide 9^2 by 27, get 3. Half (27 – 3) is 12

They meet 12 cubits from the snake's hole.

Same as the bambu sum.
Just turned around.

151) Bhuja & Difference (of other arms) given

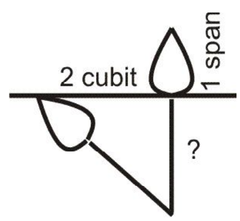
The quotient of the square of the bhuja divided by the difference between the karna and coti is twice set down, and the difference is subtracted from the quotient [in one place] and added to it [in the other]. The moieties [of the remainder and sum] are in their order the coti and karna.

Given: bhuja, diff (bet karna and coti)

Divide *'bhuja^2'* by *'diff'*
Halve its sum with 'diff' (Karna)
Halve its diff. with 'diff' (Coti)

153) Example

In a certain lake swarming with ruddy geese and cranes, the tip of a bud of lotus was seen a span above the surface of the water. Forced by the wind, it gradually advanced, and was submerged at the distance of two cubits. Compute quickly, mathematician, the depth of water.



Statement: bhuja, 4 span (2 cubits), diff: 1 span, Depth ?

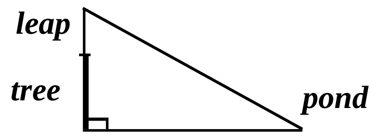
Divide sq of bhuja by diff, 16;

Half of (16 – 1), $\frac{15}{2}$ spans.

Depth of water, $\frac{15}{2}$ spans

154) Bhuja, Lower portion of Coti, Sum of 'Upper & Karna' given

The height of a tree, multiplied by its distance from the pond, is divided by twice the height of the tree added to the space between the tree and pond: the quotient will be the measure of the leap.



Given: ***tree***, ***bhuja*** (space bet tree & pond), *'leap* + *karna'* = *'tree* + *bhuja'*

Divide **prod** of '*tree* & *bhuja'* by **sum** 'twice *tree'* & *'base'*

155) Example

From a tree a hundred cubits high, an ape descended and went to a pond two hundred cubits distant: while another ape, vaulting to some height off the tree, proceeded with velocity diagonally to the same spot. If the space travelled by them be equal, tell me, quickly, learned man, the height of the leap, if thou have diligently studied calculation.

Given: tree 100, distance 200, leap ?

Prod of *tree* & *base*, 20000; Sum of twice *tree* + *base*, 400;

Dividing, 50 cubits

Height of leap **50** cubits

156) Karna & Sum (of Diff) of other Two known

From twice the square of the karna subtract the sum of the (coti and bhuja) multiplied by itself, and extract the square-root of the remainder. Set down the sum twice, and let the root be subtracted in one place and added in the other. The moities will be measures of the bhuja and coti.

157) Example

Where karna is seven above ten; the sum of the bhuja and coti, three above twenty; tell them to me, my friend.

Statement: karna 17, sum of others 23, coti ?, bhuja ?

Subt sq of sum from twice sq of karna, 578 – 529, 49
Sq rt, 7

sum – this rt, 23 – 7, 16; half 8
sum + this rt, 23 + 7, 30; half 15
bhuja 8, coti 15.

158) Example

Where the difference of the bhuja and coti is seven and diagonal is thirteen, say quickly, eminent mathematician, what are the bhuja and coti?

Statement: karna 13, diff of others 7, coti ?, bhuja ?

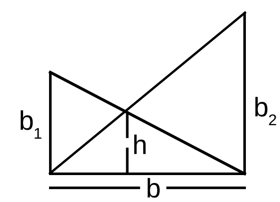
Subt sq of diff from twice sq of karna, 338 – 49, 289
Sq rt, 17

Diff of diff & this rt, 10; half 5
Sum of diff & this rt, 24; half 12
bhuja 12, coti 5.

159) Two Bambus: Height of Intersection

The product of two erect bambus being divided by their sum, the quotient is the perpendicular from the junction [intersection] of threads passing reciprocally from the root [of one] to the tip [of the other.] ...

GANESA: This perpendicular is independent of the ground



Given: Two vertical bamboos.
Strings are tied from each foot to the top of the other.
These strings cross at a point.

- The height of this point above the ground is fixed, irrespective of the distance between the bamboos.

- This height is $\boxed{\dfrac{\textit{product of bamboos}}{\textit{sum of bamboos}}}$

160) Example ...

Tell the perpendicular drawn from the intersection of strings stretched mutually from the roots to the summits of two bambus fifteen and ten cubits high standing upon gruound of unknown extent.

Bambus 15, 10.

Perpendicular = $\dfrac{15 \times 10}{15 + 10}$ = 6 cubit.